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HUYGHENS, BOHR, RIEMANN AND GALOIS: PHASE-LOCKING

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Several mathematical views of phase-locking are developed. The classical Huyghens approach is generalized to include all harmonic and subharmonic resonances and is found to be connected to $1/f$ noise and prime number theory. Two types of quantum phase-locking operators are defined, one acting on the rational numbers, the other on the elements of a Galois field. In both cases we analyse in detail the phase properties and find them related respectively to the Riemann zeta function and to incomplete Gauss sums.

Keywords: Phase-Locking; $1/f$ noise; quantum complementarity; phase states; prime numbers; cyclotomic field; Galois fields, incomplete Gauss sums.

1. Introduction

In the crude sense phase-locking occurs whenever the erratic behavior of one single piece shifts to the ordered behavior of the whole system. There is a huge number of phase-locked systems (populations of crickets, yeast cells, lasers ...) and none universal mechanism should be expected. The concept of phase pervades the whole physics and we will show that its mathematical counterpart touches several intriguing open problems.

Working experimentally it was found that the interleaving of frequencies and phases of electronic oscillators interacting in non linear circuits follows arithmetical rules. Continued fraction expansions, prime number decompositions and related number theoretical concepts were successfully used to account for the experimental effects in mixers and phase-locked loops.^{1,2} We also made use of these tools within the field of quantum optics emphasizing the hidden connection between phase-locking and cyclotomy.³ Finally a class of optimal states in quantum information happens to be quantum phase states constructed (phase-locked) from Galois fields and rings.⁴ In this paper we show that their properties are related to incomplete Gauss sums.

2. Classical Phase-Locking: from Huyghens to the Prime Numbers

Being obliged to stay in my room for several days and also occupied in making

observations on my two newly made clocks, I have noticed a remarkable effect which no one could have ever thought of. It is that these two clocks hanging next to one another separated by one or two feet keep an agreement so exact that the pendulums invariably oscillate together without variation. After admiring this for a while, I finally figured out that it occurs through a kind of sympathy: mixing up the swings of the pendulums, I have found that within a half hour always return to consonance and remain so constantly afterwards as long as I let them go. I then separated them, hanging one at the end of the room and the other fifteen feet away, and noticed that in a day there was five seconds difference between them. Consequently, their earlier agreement must in my opinion have been caused by an imperceptible agitation of the air produced by the motion of the pendulums.

The citation is taken from Ref. 5. The authors cite a later letter by Huyghens that the coupling mechanism was in fact a small vibration transmitted through the wall, and not movement of air:

Lord Rayleigh (1907) made similar observations about two driven tuning forks coupled by vibrations transmitted through the table on which both forks sat... Locking in triode circuits was explained by Van der Pol (1927) who included in the equation for the triode oscillator an external electromotive force as given in

$$\frac{d^2v}{dt^2} - \frac{d}{dt}(gv - \beta'v^3) + \omega^2v = \omega_0^2V_0 \sin \omega_0 t, \quad (1)$$

where g is the linear net gain (i.e. the gain in excess of losses, β' the saturation coefficient, and ω is the resonance frequency in the absence of dissipation or gain. He showed that when an external electromotive force is included, of frequency ω_0 , and tuned close to the oscillator frequency ω , the oscillator suddenly jumped to the external frequency. It is important to note that the beat note between the two frequencies vanishes not because the two frequencies vanish, not because the triode stops oscillating, but because it oscillates at the external frequency.

We can show the locking effect by utilizing the slowly varying amplitude approach, including a slowly varying phase Φ and oscillation at the external frequency ω_0 and amplitude V

$$\frac{d\Phi}{dt} + K \sin \Phi = \omega - \omega_0 = \omega_{LF}, \quad (2)$$

where we use ω_{LF} for the detuning term and $K = \omega_0 V_0 / V$ for the locking coefficient.⁵

The regime just described is the so-called injection locking regime, also found in injection-locked lasers. The equation (2) is the so-called Adler's equation of electronics.⁶

One way to synthesize (2) is through to the phase-locked loop of a communication receiver. The receiver is designed to compare the information carrying external oscillator (RF) to a local oscillator (LO) of about the same high frequency through a non linear mixing element. For narrow band demodulation one uses a discriminator

of which the role is first to differentiate the signal, that is to convert frequency modulation (FM) to amplitude modulation (AM) and second to detect its low frequency envelope: this is called baseband filtering. For more general FM demodulation one uses a low pass filter instead of the discriminator to remove the high frequency signals generated after the mixer. In the closed loop operation a voltage controlled LO (or VCO) is used to track the frequency of the RF. Phase modulation is frequently used for digital signals because low bit error rates can be obtained despite poor signal to noise ratio in comparison to frequency modulation.⁷

Let us consider a type of receiver which consists in a mixer, in the form of a balanced Schottky diode bridge and a low pass filter. If f_0 and f are the frequencies of the RF and the LO, and $\theta(t)$ and $\psi(t)$ their respective phases, the set mixer and filter essentially behaves as a phase detector of sensitivity u_0 (in Volts/rad.), that is the instantaneous voltage at the output is the sine of the phase difference at the inputs

$$u(t) = u_0 \sin(\theta(t) - \psi(t)). \quad (3)$$

The nonlinear dynamics of the set-up in the closed loop configuration is well described by introducing the phase difference $\Phi(t) = \theta(t) - \psi(t)$. Using $\dot{\theta} = \omega_0$ and $\dot{\psi}(t) = \omega + Au(t)$, with $\omega_0 = 2\pi f_0$, $\omega = 2\pi f$ and A (in rad. Hz/Volt) as the sensitivity of the *VCO*, one recovers Adler's equation (2) with the open loop gain $K = u_0 A$. Such a set up is called a phase-locked loop (or PLL).

Equation (2) is integrable but its solution looks complex.⁸ If the frequency shift ω_{LF} does not exceed the open loop gain K , the average frequency $\langle \dot{\Phi} \rangle$ vanishes after a finite time and reaches the stable steady state $\Phi(\infty) = 2l\pi + \sin^{-1}(\omega_{LF}/K)$, l integer. In this phase-tracking range of width $2K$ the *RF* and the *LO* oscillators are also frequency-locked. Outside the mode-locking zone there is a sech shape beat signal of frequency

$$\tilde{\omega}_{LF} = \langle \dot{\Phi}(t) \rangle = (\omega_{LF}^2 - K^2)^{1/2}. \quad (4)$$

The sech shape signal and the nonlinear dependance on parameters ω_{LF} and K are actually found in experiments.^{8,2} In addition the frequency ω_{LF} is fluctuating (see Fig. 1). It can be characterized by the Allan variance $\sigma^2(\tau)$ which is the mean squared value of the relative frequency deviation between adjacent samples in the time series, averaged over an integration time τ . Close to the phase-locked zone the Allan deviation is

$$\sigma(\tau) = \frac{\sigma_0 K}{\tilde{\omega}_{LF}}, \quad (5)$$

where σ_0 is a residual frequency deviation depending of the quality of input oscillators and that of the phase detector. Allan deviation is found independent of τ which is a signature of a $1/f$ frequency noise of power spectral density $S(f) = \sigma/(2 \ln 2f)$. One way to predict the dependence (5) is to use differentiation of (4) with respect to the frequency shift $\tilde{\omega}_{LF}$ so that

$$\delta \tilde{\omega}_{LF} = \delta \omega_{LF} (1 + K^2/\tilde{\omega}_{LF}^2)^{1/2}. \quad (6)$$

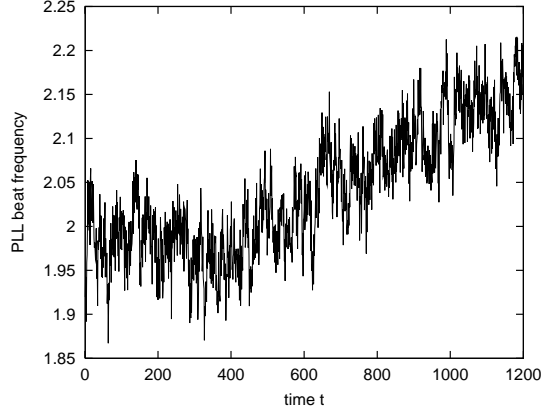


Fig. 1. Fluctuating counts of the beat frequency (in Hz) close to the phase-locked zone. The inputs are quartz oscillators at 10 MHz. The power spectrum has a pure $1/f$ dependance

Relation (6) is defined outside the mode-locked zone $|\omega_{LF}| > K$; close to it, if the effective beat note $\tilde{\omega}_{LF} \leq K$, the square root term is about $K/\tilde{\omega}_{LF}$. If one identifies $\delta\omega_{LF}/\tilde{\omega}_{LF}$ with a bare Allan deviation σ_0 and $\delta\tilde{\omega}_{LF}/\tilde{\omega}_{LF}$ with a magnified Allan deviation σ one explains the experimental result (5). One can conclude that, either the PLL set-up behaves as a microscope of an underlying flicker floor σ_0 , or the $1/f$ noise is some dynamical property of the PLL. In the past we looked at a possible low dimensional structure of the time series and found a stable embedding dimension lower or equal to 4.⁹ But at that time the dynamical model of $1/f$ noise still remained elusive.

Adler's model presupposes a fundamental interaction $\omega_{LF} = |\omega_0 - \omega(t)|$ in the mixing of the two input oscillators. But the practical operation of the phase detector involves harmonic interactions of the form $\omega_{LF} = |p\omega_0 - q\omega(t)| \leq \omega_c = 2\pi f_c$, where p and q are integers and f_c is the cut-off frequency of the low pass filter. This can be rewritten by introducing the frequency ratios $\nu = \frac{\omega(t)}{\omega_0}$ and $\mu = \frac{\omega_{LF}}{\omega_0}$ as $\mu = q|\nu - \frac{p}{q}|$. This form suggests that the aim of the receiver is to select such pairs (p, q) which realize a “good” approximation of the “real” number ν . There is a mathematical concept which precisely does that: the diophantine approximator. It selects such pairs p_i and q_i , coprime to each other, i.e. with greatest common divisor $(p_i, q_i) = 1$ from the continued fraction expansion of ν truncated at the index i

$$\nu = \{a_0; a_1, a_2, \dots, a_i\} = a_0 + 1/(a_1 + 1/(a_2 + 1/\dots + 1/a_i)) = \frac{p_i}{q_i}. \quad (7)$$

The diophantine approximation satisfies

$$|\nu - \frac{p_i}{q_i}| \leq \frac{1}{a_{i+1}q_i^2}. \quad (8)$$

The fraction $\frac{p_i}{q_i}$ is a so-called convergent and the a_i 's are called partial quotients. The approximation is truncated at the index i just before the partial quotient a_{i+1} .

It should be observed that diophantine approximations are different from decimal approximations $\frac{c_i}{d_i}$ for which one gets $|\nu - \frac{c_i}{d_i}| \leq \frac{1}{d_i}$. It was shown in Ref. 1, using the filtering condition, that a_{i+1} is given by

$$a_{i+1} = \left[\frac{f_0}{f_c q_i} \right], \quad (9)$$

where $[]$ denotes the integer part. For example if one chooses $f_0 = 10$ MHz and $f_c = 300$ kHz, the fundamental basin $\frac{p_i}{q_i} = \frac{1}{1}$ will be truncated if $a_{i+1} \geq 33$ and the basin $\frac{p_i}{q_i} = \frac{3}{5}$ will be truncated if $a_{i+1} \geq 6$. The resulting full spectrum is a superposition of V-shape basins of which the edges are located at

$$\begin{aligned} \nu_1 &= \{a_0; , a_1, a_2, \dots, a_i, a_{i+1}\}, \\ \nu_2 &= \{a_0; a_1, a_2, \dots, a_{i-1}, 1, a_{i+1}\}, \end{aligned} \quad (10)$$

where the partial expansion before a_{i+1} corresponds to the two possible continued fractions of the rational number $\frac{p_i}{q_i}$. The basin of number $\nu = \frac{3}{5} = \{0; 1, 1, 2\}$ extends to $\nu_1 = \{0; 1, 1, 2, 33\} = \frac{19}{32} \simeq 0.594$, $\nu_2 = \{0; 1, 1, 1, 1, 33\} = \frac{31}{34} \simeq 0.618$. For a reference oscillator with $f_0 = 10$ MHz this corresponds to a frequency bandwidth $(0.618 - 0.594) \cdot 10^7$ MHz = 240 kHz.

With these arithmetical rules in mind one can now tackle the difficult task of accounting for phase-locking of the whole set of harmonics in the beat frequency.²

$$\omega_{LF} = |p_i \omega_0 - q_i \omega(t)|, \quad (11)$$

Some essential features can be found in the standard Arnold map model²

$$\Phi_{n+1} = \Phi_n + 2\pi\Omega - c \sin \Phi_n, \quad (12)$$

where $\Omega = \frac{\omega}{\omega_0}$ is the bare frequency ratio and $c = \frac{K}{\omega_0}$. Such a nonlinear map is studied by introducing the winding number $\nu = \lim_{n \rightarrow \infty} (\Phi_n - \Phi_0)/(2\pi n)$. The limit exists everywhere as long as $c < 1$, the curve ν versus ω is a devil's staircase with steps attached to rational values $\Omega = \frac{p_i}{q_i}$ and width increasing with the coupling coefficient c . The phase-locking zones may overlap if $c > 1$ leading to chaos from quasi-periodicity.¹⁰

The Arnold map is also a relevant model of a short Josephson junction shunted by a strong resistance R and driven by a periodic current of frequency ω_0 and amplitude I_0 . Steps are found at the driving voltages $V_r = RI_0 = r(\hbar\omega_0/2e)$, r a rational number. Fundamental resonances $r = n$, n integer, have been used to achieve a voltage standard of relative uncertainty 10^{-7} .

To appreciate the impact of harmonics on the coupling coefficient one may observe that each harmonic of denominator q_i creates the same noise contribution $\delta\omega_{LF} = q_i \delta\omega(t)$. They are $\phi(q_i)$ of them, where $\phi(q_i)$ is the Euler totient function, that is the number of integers less or equal to q_i and prime to it; the average coupling coefficient is thus expected to be $1/\phi(q_i)$. In Ref. 2 a more refined model is developed based on the properties of prime numbers. It is based on defining a coupling

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coefficient as $c^* = c\Lambda(n; q_i, p_i)$ with $\Lambda(n; q_i, p_i)$ a generalized Mangoldt function.² It is defined as

$$\Lambda(n; q_i, p_i) = \begin{cases} \ln b & \text{if } n = b^k, \text{ } b \text{ a prime and } n \equiv p_i \pmod{q_i}, \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

The classical Mangoldt function is $\Lambda(n) = \Lambda(n; 1, 1)$. It is thus the coupling coefficient of the fundamental resonance $1/1$. The important result of that analysis is to exhibit a fluctuating average coefficient as follows

$$c_{\text{av}}^*/c = \frac{1}{t} \sum_{n=1}^t \Lambda(n; q_i, p_i) = \frac{1}{\phi(q_i)} + \epsilon(t), \quad (14)$$

with $\epsilon(t) = O(t^{-1/2} \ln^2(t))$ which is known to be a good estimate as long as $q_i < \sqrt{t}$.¹ The average coupling coefficient shows the expected dependance on q_i . In addition there is an arithmetical noise $\epsilon(t)$ with a low frequency dependance of the power spectrum reminding $1/f$ noise. Although that stage of the theory is not the last word of the story, it is quite satisfactory that this approach, based on phase-locking of the full set of harmonics, is accounting for the main aspects of $1/f$ noise found in experiments.

3. Quantum Phase-Locking

Apparently^a Dirac was the first to attempt a definition of a phase operator by means of an operator amplitude and phase decomposition. As we have discussed, with a complex c -number $a = Re^{i\Phi}$ one obtains the phase via $e^{i\Phi} = a/R$. Similarly, he sought to decompose the annihilation operator a into amplitude and phase components... After a brief calculation we obtain a relation indicating that the number operator N and phase operator Φ are canonically conjugate

$$[N, \Phi] = 1. \quad (15)$$

The equation immediately leads to a number-phase uncertainty relation which is often seen

$$\delta N \delta \Phi \geq 1/2. \quad (16)$$

However, all of the previous development founders upon closer examination.

This is taken from Ref. 11, a comprehensive review of the quantum phase problem. See also Ref. 13.

To approach the phase-locking problem within quantum mechanics one can start from the theory of the harmonic oscillator. The natural objects are the Fock states (the photon occupation states) $|n\rangle$ who live in an infinite dimensional Hilbert space. They are orthogonal to each other: $\langle n|m\rangle = \delta_{mn}$, where δ_{mn} is the Dirac symbol. The states form a complete set: $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$.

^aOne referee nominated Madelung¹² as the first investigator of quantum phase operators.

The annihilation operator removes one photon from the electromagnetic field

$$a|n\rangle = \sqrt{n}|n-1\rangle, n = 1, 2, \dots \quad (17)$$

Similarly the creation operator a^\dagger adds one photon: $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $n = 0, 1, \dots$. There is the commutation relation $[a, a^\dagger] = 1$. The operator $N = aa^\dagger$ has the meaning of the particle number operator and satisfies the eigenvalue equation $N|n\rangle = n|n\rangle$.

Eigenvectors of the annihilation operator are the so-called coherent states $|\alpha\rangle$ ¹⁴ and are the ones generated by single mode laser operated well above threshold.

States of well defined phase escaping the inconsistencies of Dirac's formulation were built by Susskind and Glogower.¹⁵ They correspond to the eigenvalues of the exponential operator

$$E = e^{i\Phi} = (N+1)^{-1/2}a = \sum_{n=0}^{\infty} |n\rangle\langle n+1|. \quad (18)$$

Using the Hermitian conjugate operator $E^\dagger = e^{-i\Phi}$, one gets $EE^\dagger = 1$, $E^\dagger E = 1 - |0\rangle\langle 0|$, i.e. the unitarity of E is spoiled by the vacuum-state projector $|0\rangle\langle 0|$. The Susskind-Glogower phase states satisfy the eigenvalue equation $E|\Psi\rangle = e^{i\psi}|\Psi\rangle$; they are given as

$$|\Psi\rangle = \sum_{n=0}^{\infty} e^{in\psi}|n\rangle. \quad (19)$$

Like the coherent states the phase states are non orthogonal and they form an overcomplete basis which solves the identity operator: $\frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi |\psi\rangle\langle\psi| = 1$. The operator $\cos\Phi = \frac{1}{2}(E + E^\dagger)$ is used in the theory of Cooper pair box with a very thin junction when the junction energy $E_J \cos\Phi$ is higher than the electrostatic energy.¹⁶

Further progress in the definition of phase operator was obtained by Pegg and Barnett.¹⁸ The phase states are defined from the discrete Fourier transform (or more precisely the quantum Fourier transform since the superposition is on Fock states not on real numbers)

$$|\theta_k\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp(2i\pi \frac{k}{q}n) |n\rangle. \quad (20)$$

The states are eigenstates of the Hermitian phase operator

$$\Theta_q = \sum_{k=0}^{q-1} \theta_k |\theta_k\rangle\langle\theta_k|, \quad (21)$$

with $\theta_k = \theta_0 + 2\pi k/q$ and θ_0 is a reference angle. It is implicit in the definition (20) that the Hilbert space is of finite dimension q . The states $|\theta_k\rangle$ form an orthonormal set and in addition the projector over the subspace of phase states is $\sum_{k=0}^{q-1} |\theta_k\rangle\langle\theta_k| = 1_q$, where 1_q is the unity operator. Given a state $|F\rangle$ one can

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write a probability distribution $|\langle \theta_k | F \rangle|^2$ which may be used to compute various moments, e.g. expectation values, variances. The key element of the formalism is that first the calculations are done in the subspace of dimension q , then the limit $q \rightarrow \infty$ is taken.¹⁸

We are now in position to define a quantum phase-locking operator.¹⁷ Our viewpoint has much to share with the classical phase-locking problem as soon as one reinterpret the fraction $\frac{k}{q}$ in (20) as arising from the resonant interaction between two oscillators and the denominator q as a number which defines the resolution of the experiment. From now we emphasize such phase states $|\theta'_k\rangle$ which satisfy phase-locking properties and we impose the coprimality condition

$$(k, q) = 1. \quad (22)$$

The quantum phase-locking operator is defined as

$$\Theta_q^{\text{lock}} = \sum_k' \theta_k |\theta'_k\rangle \langle \theta'_k|, \quad (23)$$

with $\theta_k = 2\pi \frac{k}{q}$ and the notation \sum' means summation from 0 to $q-1$ with $(k, q) = 1$. Using (20) and (22) in (23) one obtains

$$\Theta_q^{\text{lock}} = \frac{1}{q} \sum_{n,l} c_q(n-l) |n\rangle \langle l|, \quad (24)$$

where the range of values of n, l is from 0 to $\phi(q)$, and $\phi(q)$ is the Euler totient function. The coefficients in front of the outer products $|n\rangle \langle l|$ are the so-called Ramanujan sums

$$c_q(n) = \sum_k' \exp(2i\pi \frac{kn}{q}) = \frac{\mu(q_1)\phi(q)}{\phi(q_1)}, \quad \text{with } q_1 = q/(q, n). \quad (25)$$

In the above equation $\mu(q)$ is the Möbius function, which is 0 if the prime number decomposition of q contains a square, 1 if $q = 1$ and $(-1)^K$ if q is the product of K distinct primes. Ramanujan sums are relative integers which are quasi-periodic versus n with quasi-period $\phi(q)$, and aperiodic versus q with a type of variability imposed by the Möbius function. Ramanujan sums have been used for signal processing of low frequency noise.¹⁹ In the Ramanujan sum expansion there is a modified Mangoldt function $b(n)$ which is the dual of Möbius function

$$b(n) = \frac{\phi(n)}{n} \Lambda(n) = \sum_{q \geq 1} \frac{\mu(q)}{\phi(q)} c_q(n). \quad (26)$$

This illustrates that many “interesting” arithmetical functions carry the structure of prime numbers. We mention the relation $d \ln \zeta(s)/ds = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}$, but there is also the relation $1/\zeta(s) = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$. There is a well known formulation¹ of Riemann hypothesis from the summatory Möbius function $\sum_{n=1}^t \mu(n) = O(t^{1/2+\epsilon})$, whatever ϵ .

Given a state $|\beta\rangle$ one can calculate the expectation value of the quantum phase-locking operator as

$$\langle \Theta_q^{\text{lock}} \rangle = \sum_k \theta_k |\langle \theta'_k | \beta \rangle|^2. \quad (27)$$

If one uses the finite form of Susskind-Glogower phase states (19) and a real parameter β

$$|\beta\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \exp(in\beta) |n\rangle, \quad (28)$$

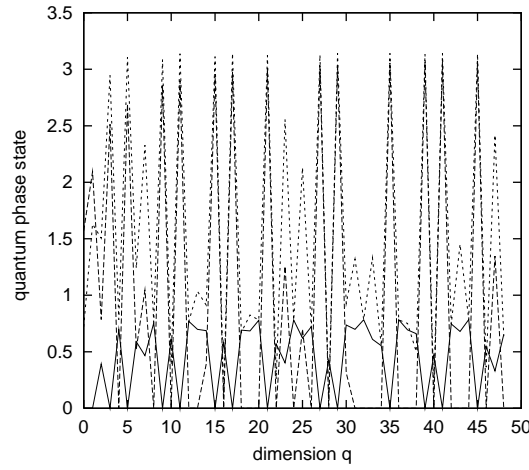


Fig. 2. Oscillations in the expectation value (29) of the locked phase at $\beta = 1$ (dotted line) and their squeezing at $\beta = 0$ (plain line). The brokenhearted line which touches the horizontal axis is $\pi\Lambda(q)/\ln q$

the expectation value of the locked phase becomes

$$\langle \Theta_q^{\text{lock}} \rangle = \frac{\pi}{q^2} \sum_{n,l} c_q(l-n) \exp(i\beta(n-l)). \quad (29)$$

For $\beta = 1$ it is found that $\langle \Theta_q^{\text{lock}} \rangle$ has the more pronounced peaks are at such values of q which are powers of a prime number. It can be approximated by the normalized Mangoldt function $\pi\Lambda(q)/\ln q$ as shown on Fig. 2. For $\beta = 0$ the expectation value of $\langle \Theta_q^{\text{lock}} \rangle$ is much lower. The parameter β can be used to minimize the phase uncertainty well below the classical value.³

Quantum phase-locking effect and its relation to prime number theory has also been studied implicitly by Bost and Connes.²⁰ Instead of an ad-hoc quantum phase operator as (21) or (24), it is based on the formulation of a dynamical system

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and its associated quantum statistics. The dynamical system is first defined by an Hamiltonian operator H_0 with eigenvalues equal to the logarithms of integers

$$H_0|n\rangle = \ln n|n\rangle. \quad (30)$$

Using the relations $\exp(-\beta_0 H_0)|n\rangle = \exp(-\beta_0 \ln n)|n\rangle = n^{-\beta_0}|n\rangle$, it follows that the partition function of the model at the inverse temperature β_0 is

$$\text{Trace}(\exp(-\beta_0 H_0)) = \sum_{n=1}^{\infty} n^{-\beta_0} = \zeta(\beta_0), \quad (31)$$

where $\zeta(\beta_0)$ is the Riemann zeta function

In quantum statistical mechanics, given an observable Hermitian operator M one has the Hamiltonian evolution $\sigma_t(M)$ versus time t

$$\sigma_t(M) = e^{itH_0} M e^{-itH_0}, \quad (32)$$

and the expectation value of M is the Gibbs state

$\text{Gibbs}(M) = \text{Trace}(M \exp(-\beta_0 H_0)) / \text{Trace}(\exp(-\beta_0 H_0))$. In Bost and Connes ap-

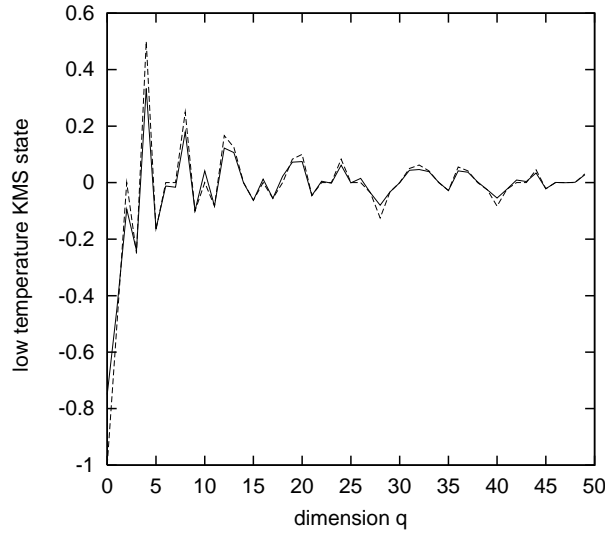


Fig. 3. Phase expectation value (35) in Bost and Connes model at the inverse temperature $\beta = 3$ (plain line) in comparison to the function $\mu(q)/\phi(q)$ (dotted line)

proach the observables belong to an algebra of operators μ_a and e_k which are defined by their action on the occupation numbers $|n\rangle$ as

$$\mu_a|n\rangle = |an \pmod{q}\rangle, \quad (33)$$

$$e_k|n\rangle = \exp\left(\frac{2i\pi kn}{q}\right)|n\rangle. \quad (34)$$

The first operator μ_a acts as a shift in the space of number states; the second one e_k is such that its action encodes the individuals in the quantum Fourier transform

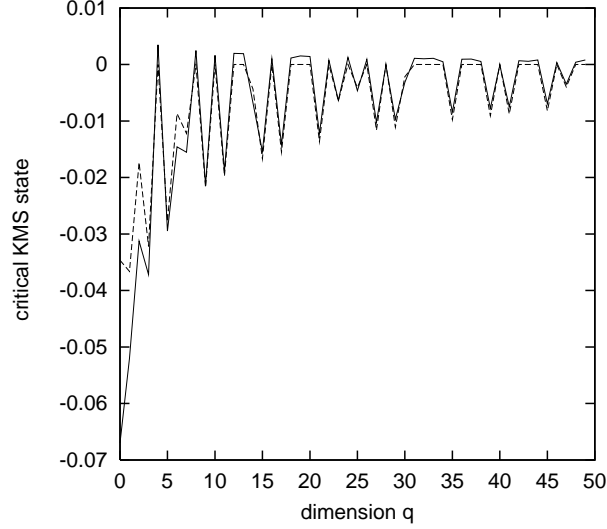


Fig. 4. Phase expectation value (35) in Bost and Connes model at the inverse temperature $\beta = 1 + \epsilon$, $\epsilon = 0.1$ (plain line) in comparison to the function $-\Lambda(q)\epsilon/q$ (dotted line).

(20). Like in the quantum phase-locking operator one uses the coprimality condition (22) to distinguish in (34) the primitive roots of unity $\exp(2i\pi k/q)$, $(k, q) = 1$. One can show that there is a hidden symmetry group which is used to label the elements of the algebra^b. Using the action of the group, the Gibbs state is replaced by the so-called Kubo-Martin-Schwinger (or KMS) state. The system exhibits a phase transition with spontaneous symmetry breaking at the inverse temperature $\beta_0 = 1$ which corresponds to the unique pole of the Riemann zeta function $\zeta(\beta_0)$. At low temperature $\beta_0 > 1$ one gets, after tricky calculations, the expectation value of the phase operator which replaces (29) in the following form²⁰

$$\text{KMS}(e_q^{(p)}) = q^{-\beta_0} \prod_{\substack{p \text{ divides } q \\ p \text{ prime}}} \frac{1 - p^{\beta_0 - 1}}{1 - p^{-1}}. \quad (35)$$

The KMS state is represented for two limiting cases, the low temperature limit $\beta \gg 1$ (Fig. 3) and the critical case $\beta = 1 + \epsilon$ (Fig. 4), with $\epsilon \simeq 0$. In these limits one has respectively $\text{KMS}_{\beta \gg 1}(q) = \frac{\mu(q)}{\phi(q)}$ and $\text{KMS}_{1+\epsilon} \simeq \frac{-\Lambda(q)\epsilon}{q}$. In the low temperature limit the spectrum (26) corresponding to the Ramanujan sum expansion of the modified Mangoldt function $b(n) = \frac{\Lambda(n)\phi(n)}{n} = \Lambda(n)$ is recovered (see (26)). Close to the critical point $\beta = 1 + \epsilon$ the oscillations are proportional to $\Lambda(q) \simeq b(q)$ and are of very small amplitude due to the squeezing coefficient ϵ . A comparable squeezing effect was already observed in the expectation value (27) of the quantum phase

^bThis is the Galois group $W = \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ of the cyclotomic extension on the field of rational numbers \mathbb{Q} .

operator (see Fig. 2).

Thus after the phenomenological model (14), the Bost and Connes cyclotomic model also points to the Mangoldt function as a source of low frequency fluctuations. In the last case the model is associated to the spontaneous symmetry breaking and the squeezing of phase oscillations at the critical KMS state ^c.

4. Galois Phase-locking and Quantum Complementarity

But what is light really? Is it a wave or a shower of photons? There seems no likelihood for forming a consistent description of the phenomena of light by a choice of only one of the two languages. It seems as though we must use sometimes the one theory and sometimes the other, while at times we may use either. We are faced with a new kind of difficulty. We have two contradictory pictures of reality; separately neither of them fully explains the phenomena of light, but together they do."

(Albert Einstein and Leopold Infeld, *The Evolution of Physics*, p. 262).

More hints into quantum phase-locking may be discovered by deriving a mathematical view of the complementarity principle. At the conceptual level, two observables are complementary if precise knowledge of one of them implies that all possible outcomes of measuring the other one are equally probable. The eigenstates of such complementary observables are non-orthogonal quantum states, and in any attempt to distinguish between them, information gain is only possible at the expense of introducing disturbance.

Mathematically speaking let \mathcal{O} be an observable in a Hilbert space of dimension q , \mathcal{H}_q , which is represented by a Hermitian $q \times q$ matrix. Let us assume that its real eigenvalues are multiplicity-free and its eigenvectors $|b\rangle$ belong to an orthonormal basis B . Let \mathcal{O}' be a (prepared) complementary observable with eigenvectors $|b'\rangle$ in B' . If \mathcal{O} is measured, then the probability to find the system in the state $|b\rangle \in B$ is given by $|\langle b|b'\rangle|^2 = 1/q$. We here recall that two orthonormal bases B and B' of \mathcal{H}_q are mutually unbiased precisely when $|\langle b|b'\rangle|^2 = \frac{1}{q}$ for all $b \in B$ and $b' \in B'$. It can be shown that in order to fully recover the density matrix of a set of identical copies of a quantum state, we need at least $q + 1$ measurements performed on complementary observables. As a matter of fact, the mathematical implementation of the complementary principle lead us to the search of complete sets of mutually unbiased bases (or MUBs for short), a problem which has recently received a peculiar attention.²²

In dimension $q = 2$, eigenvectors of ordinary Pauli spin matrices (i.e. in dimension $q = 2$) provide the best known example of a complete set of MUBs. It has been shown that in dimension $q = p^m$ which is the power of a prime p , the complete sets of mutually MUBs result from Fourier analysis over a Galois field \mathbf{F}_q (in odd characteristic p)²³ or of a Galois ring \mathbf{R}_{4^m} (when $p = 2$) (see Refs. 24 and 4 for more details). Complete sets of MUBs have an intrinsic geometrical interpretation, and

^cThe theory was also used as a model of time perception.²¹

were for example related to discrete phase spaces²⁵ or finite projective planes.^{26,27}

The complete sets of MUBs in odd power dimension q have a very compact Fourier form⁴

$$|\theta_b^a\rangle = \frac{1}{\sqrt{q}} \sum_{n \in \mathbf{F}_q} \psi(n) \kappa(an^2 + bn) |n\rangle, \quad a, b \in \mathbf{F}_q. \quad (36)$$

in which the coefficient in the computational base $\{|0\rangle, |1\rangle, \dots, |q-1\rangle\}$ represents the product of an arbitrary multiplicative character $\psi(n)$ by an arbitrary additive character $\kappa(y)$, and where the decomposition $y = an + b$, has been used.

Let fix a primitive root g in \mathbf{F}_q , then every $n \in \mathbf{F}_q$ is given by $n = g^s$ for some $s \in [0, q-2]$ and then

$$\psi(n) = \omega_{q-1}^{ks}, \quad (37)$$

is a multiplicative character. Every multiplicative character can be obtained in this way.

The additive characters are defined as

$$\kappa(x) = \omega_p^{tr(x)}, \quad \omega_p = \exp\left(\frac{2i\pi}{p}\right), \quad x \in \mathbf{F}_q. \quad (38)$$

where $tr(x) = x + x^p + \dots + x^{p^{m-1}}$ is the field theoretical trace. It maps any element of \mathbf{F}_q to an element in the base field \mathbf{F}_p .

Eq. (36) defines a set of q bases (with index a) of q vectors (with index b). Using a property of Weil sums⁴ it is easily shown that, for q odd, the bases are orthogonal and mutually unbiased to each other and to the computational base.

The result of Wootters and Field corresponds to the trivial multiplicative character $\psi_0 = 1$. Eq. (36) also defines phase states generalizing those written before in (20). The latter are recovered if one uses the trivial additive character $\kappa = \kappa_0$ in (36)^d.

4.1. The phase expectation value

For the evaluation of the phase properties of a general pure state of an electromagnetic field mode in the Galois number field we proceed as in Sect. 3. We consider the pure state of the form (28), and we sketch the computation of the probability distribution $S = |\langle \theta_b | \beta \rangle|^2$ and phase expectation value $\langle \Theta_{\text{Gal}} \rangle = \sum_{b \in \mathbf{F}_q} \theta_b |\langle \theta_b | \beta \rangle|^2$. We recall that $\theta_b = 2\pi b/q$ (the upper index a for the base is implicit and we discard it for simplicity). The probability distribution reads explicitly as⁴

$$\frac{1}{q^2} \left[\sum_{n \in \mathbf{F}_q} \psi(-n) \exp(in\beta) \kappa(-an^2 - bn) \right] \left[\sum_{m \in \mathbf{F}_q} \psi(m) \exp(-im\beta) \kappa(am^2 + bm) \right]. \quad (39)$$

^dIn (20) we used the (non prime) integer p instead of k .

Strictly speaking the phase state defined in (28) is defined only in prime dimension $q = p$. In non-prime dimensions the product in the exponential $n\beta$ of (28) is not in \mathbf{F}_p .

For prime dimensions p the phase probability distribution reads

$$\begin{aligned} S &= \frac{1}{p^2} \sum_{n=1}^p \psi(n) \exp(2i\pi(\gamma n + 2an/p)) \\ &\quad \times \sum_{m=1}^p \bar{\psi}(m) \exp(-2i\pi(\gamma m + 2am/p)) \\ &= \frac{1}{p^2} \sum_{n,m=1}^p \psi(n) \bar{\psi}(m) \exp(2i\pi(\gamma(n-m) + a(n-m)(n+m)/p)) \\ &= \frac{1}{p^2} \sum_{n=1}^p \sum_{k=n-p}^{n-1} \psi(n) \bar{\psi}(n+k) \exp(2i\pi\gamma k + ak(2n+k)/p) \\ &= \frac{1}{p^2} \sum_{k=-p+1}^{p-1} \exp(2i\pi\gamma k) T(k). \end{aligned}$$

where we used the notation $\gamma = -\beta/2\pi + b/p$. In the last but one equality above we put $k = n - m$ and in the last equality we changed the order of summation, pulling out the γ -dependant factor outside the n summation. The inner sum equals

$$T(k) = \sum_{n=\max\{1, 1-k\}}^{\min\{p-1, p-1-k\}} \psi(n) \bar{\psi}(n+k) \exp(2i\pi ak(2n+k)/p). \quad (40)$$

Now, if $k \equiv 0 \pmod{p}$ (which may happen only for $k = 0$ in the above range) the inner sum is trivial and is equal to p . For other values of k the inner sum is at most of absolute value $O(p^{1/2} \ln p)$ by the Weil bound of incomplete sums^{28,29} (note that the factor involving γ is now gone from the sum over n).

Hence

$$|S(k)| = \frac{1}{p^2} O(1 \cdot p + p \cdot p^{1/2} \ln p) = O(p^{-1/2} \ln p). \quad (41)$$

We get $|S| \simeq 0.63$ at $p=3$ and 0.49 at $p=7$. Then it decreases slowly with increasing dimension p .

The phase expectation value reads

$$\langle \Theta_{\text{Gal}} \rangle = \frac{2\pi}{p^3} \sum_{k=-p+1}^{p+1} \exp(-i\beta k) T(k) \sum_{b=1}^p b \exp(2i\pi kb/p). \quad (42)$$

The partial sums in the above equation can be evaluated as $p(p+1)/2$ for $k=0$ and otherwise

$$U = \sum_{b=1}^p b \epsilon^b = \epsilon(1 + 2\epsilon + 3\epsilon^2 + \dots + p\epsilon^{p-1}) = \epsilon \left[\frac{1 - \epsilon^p}{(1 - \epsilon)^2} - \frac{p\epsilon^p}{1 - \epsilon} \right] = \frac{\epsilon p}{\epsilon - 1}, \quad (43)$$

where we introduced $\epsilon = \exp(2i\pi k/p)$ and we made use of the relation $\epsilon^p = 1$. Easy calculations lead to

$$|U| = \frac{p}{2|\sin(2k\pi/p)|}. \quad (44)$$

An estimate of the phase expectation value can be obtained as follows.³⁰ Let r_k be the smallest (by absolute value) residue of $2k \pmod{p}$. Then $|\sin(2\pi k/p)| =$

$\sin(\pi|r_k|/p) \geq 2p|r_k|/\pi$ since $\sin(x) \geq 2x/\pi$ for $0 \leq x \leq \pi/2$. We now define $r_k = 1$ for $k = 0$. Thus, we now have $U(k) = O(p^2/r_k)$ for any k . Therefore $\sum_{k=-p+1}^{p-1} |U(k)| = O(p^2 \sum_{k=-p+1}^{p+1} 1/r_k)$.

When k runs between $-p-1$ and $p+1$, with $k \neq 0$, $r_k = 2k \pmod{p}$ takes each value in the range $[0, (p-1)/2]$ no more than 4 times. The contribution from $k = 0$ is simply 1. So the contribution is $O(p^2 \sum_{k=-p+1}^{p+1} 1/r_k) = O(p^2 \sum_{s=1}^{(p-1)/2} 1/s) = O(p^2 \log p)$.

As a result the phase expectation value is bounded by

$$\begin{aligned} | < \Theta_{\text{Gal}} > | &= \frac{2\pi}{p^3} O(1.p. \frac{p(p+1)}{2} + p.p^{1/2} \ln p.p^2 \ln p) \\ &= O(1 + p^{1/2}(\ln p)^2), \end{aligned} \quad (45)$$

which is a diverging quantity.

When $\psi(n)$ is a trivial character equal to 1, the estimate on incomplete Gauss sums (40) is replaced by the better bound $O(p/\min(k, p-k))$. It follows that $S = O(p^{-1} \ln p)^e$.

Here is the proof. When $\psi(n)$ is trivial we have $T(k) = \sum_{n=N}^M \exp(4i\pi akn/p)$ where N and M the lower and upper limits in (40). Assume that $k \neq 0$. Then $T(k) = (\exp(4i\pi akN/p) - \exp(4i\pi akM/p))/(1 - \exp(4i\pi ak/p))$. Estimating the numerator as 2, we get $|T(k)| \leq 2/|1 - \exp(4i\pi ak/p)| = 1/|\sin(2\pi ak/p)|$.

Let now use the notation s_k for the smallest (by absolute value) residue of $2ak \pmod{p}$. Therefore $\sum_{k=-p+1}^{p-1} |T(k)| = O(p \sum_{k=-p+1}^{p+1} 1/s_k) = O(p \ln p)$ and $S(k) = O(p^{-1} \ln p)$ as expected.

4.2. Phase variance

The phase variance can be written as

$$< \Delta \Theta_{\text{Gal}}^2 > = \sum_{b \in \mathbb{F}_q} (\theta_b - < \Theta_{\text{Gal}} >)^2 | < \theta_b | f > |^2. \quad (46)$$

For the first term one gets

$$\sum_{b=1}^p \theta_b^2 | < \theta_b | \beta > |^2 = \frac{4\pi^2}{p^4} \sum_{k=-p+1}^{p+1} \exp(-i\beta k) T(k) \sum_{b=1}^p b^2 \exp(2i\pi kb/p). \quad (47)$$

The partial sums in the above equation can be evaluated as $\frac{p^3}{3}$ for $k = 0$ and otherwise

$$V = \sum_{b=1}^p b^2 \epsilon^b = \epsilon \frac{d}{d\epsilon} \left(\frac{p}{\epsilon - 1} \right) = \frac{-p\epsilon}{(\epsilon - 1)^2} = \frac{p^3}{4 \sin^2(\pi k/p)}. \quad (48)$$

Using the same type of reasoning than in the last section

$$\frac{4\pi^2}{p^4} O(1.p. \frac{p^3}{3} + p.p^{1/2} \ln p.p^3 \ln p) = O(1 + 3p^{1/2}(\ln p)^2). \quad (49)$$

^eHowever one way to squeeze the bound on the phase expectation value to π is to project on the state $|\beta\rangle = |0\rangle$ in (28) as it was done in Fig. 2 in the context of ‘‘Ramanujan-type’’ phase states .

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The second term is

$$\langle \Theta_{\text{Gal}} \rangle^2 \sum_{b=1}^p |\langle \theta_b | \beta \rangle|^2 = \frac{1}{p^2} \sum_{k=-p+1}^{p+1} \exp(-i\beta k) T(k) \sum_{b=1}^p \exp(2i\pi k b/p). \quad (50)$$

The inner sum equals p if $k = 0$ and 0 otherwise, so that the whole contribution is $O(1)$.

Finally the third term is

$$-2 \langle \Theta_{\text{Gal}} \rangle \sum_{b=1}^p \theta_b |\langle \theta_b | \beta \rangle|^2 = -2 \langle \Theta_{\text{Gal}} \rangle^2. \quad (51)$$

Using (46) the absolute value is bounded by

$$O(1 + p(\ln p)^4). \quad (52)$$

All estimates of the contributing terms in the variance are diverging with p .

5. Maximally entangled states

By definition entangled states in \mathcal{H}_q cannot be factored into tensorial products of states in Hilbert spaces of lower dimensions. There is an intrinsic relation between MUBs and maximal entanglement.

Generalized Bell states may be defined using the multiplicative Fourier transform (21) applied to the tensorial products of two qudits,

$$|\mathcal{B}_{u,k}\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \omega_q^{kn} |n, n+u\rangle. \quad (53)$$

Also these states are both orthonormal, $\langle \mathcal{B}_{u,k} | \mathcal{B}_{u',k'} \rangle = \delta_{uu'} \delta_{kk'}$, and maximally entangled, $\text{trace}_2 |\mathcal{B}_{u,k}\rangle \langle \mathcal{B}_{u,k}| = \frac{1}{q} I_q$. A more general class of maximally entangled states is obtained using the Fourier expansion (36) over \mathbf{F}_q (q odd) as follows

$$|\mathcal{B}_{u,b}^a\rangle = \frac{1}{\sqrt{q}} \sum_{n=0}^{q-1} \omega_p^{\text{tr}[(an+b)n]} |n, n+u\rangle. \quad (54)$$

In general, for q a power of a prime, starting from (54) one obtains q^2 bases of q maximally entangled states. Each set of the q bases (with u fixed) has the property of mutual unbiasedness. Similarly, for sets of maximally entangled m -qubits one uses the Fourier transform over Galois rings.⁴

6. Conclusion

The phase relation between a single piece and the whole system is strongly contextual, but the working mathematics is amazingly universal. In an electronic phase-locked loop we found that the position of mode-locked zones is controlled by the arithmetic of irreducible fractions (7)-(11), and the strength of lockings is related to

prime number theory via the Mangoldt function (14). We also developed two different approaches of phase-locking within the context of quantum optics. One of them uses a discrete phase operator (23), and the phase expectation value also relates to prime number theory via Ramanujan sums (see (27)). In a more sophisticated form it is linked to the Riemann zeta function (see (35)). In a second approach, the phase locked states are properly defined Fourier transforms over a Galois field (see (36)). They connect to mutually unbiased bases, which appears in the mathematical formulation of quantum complementarity. Incomplete Gauss sums are at the kernel of phase variability in this case.

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